# *Fractal-Based Point Processes*

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# 1

# *Introduction*

**Albrecht Penck (1858–1945)**, a German geographer and geologist known particularly for his studies of glaciation in the Alps, recognized that the length of a coastline depends on the scale at which it is measured.

**Lewis Fry Richardson (1881–1953)**, a British mathematician and Quaker pacifist, studied turbulence, weather prediction, the statistics of wars, and the relationship between length and measurement scales.

## 1.1 FRACTALS

What is the length of a coastline? Albrecht Penck, a Professor of Geography at the University of Vienna, observed more than a hundred years ago that the apparent length of a coastline grew larger as the size of the map increased (Penck, 1894). He concluded that this apparently simple question has a complex answer — the outcome of the measurement depends on the scale at which the coastline is measured.

Using a detailed map of a given coastline, and carefully tracing all of its bays and peninsulas, a sufficiently patient geographer could follow the features and arrive at a number for its length. A more hurried geographer, using a map of lower resolution that follows the coastline less closely, would obtain a smaller result since many of the features seen by the first geographer would be absent. In general, higher measurement precision yields a greater number of discernible details, and consequently results in a coastline of greater length. The question "What is the length of a coastline?" has no single answer.

This phenomenon is illustrated in Fig. 1.1 for a section of the Icelandic coastline between the towns of Seyðisfjörður and Höfn. The three maps, illustrated in different shades of gray, are identical; only the scale used to measure the length of the coastline differs. The measurement indicated by the white curve on the dark-gray map traces features with a scale of 0.694 km. Following all of the features of the map at this scale requires 769 segments, each of length 0.694 km, which yields an overall coastline length of 534 km. This measurement closely hugs the coastline. The medium-gray map, whose boundary (black curve) is measured with a 6.94-km scale, requires just over 45 segments, leading to a coastline length of 314 km. This coarser measurement follows the coastline more approximately; the result therefore appears more jagged and yields a shorter length. Finally, a measurement made at a scale of 69.4 km, represented on the light-gray map, yields the shortest length of the three: 133 km.

Were the scale to increase further, a minimum distance of 125 km would obtain for scales in excess of 125 km, since that is the point-to-point distance between the two towns. At the opposite extreme, if the measurement scale were to reach below 0.694 km, the length of the coastline would grow beyond 534 km, as ever smaller bays and peninsulas, rocks, and grains of sand were taken into account.

Although coastlines do not have well-defined lengths, as recognized by Penck (1894), and subsequently by Steinhaus (1954) and Perkal (1958a,b), an empirical mathematical relation connecting the measured coastline length and the measurement scale was discovered by Lewis Fry Richardson. In his *Appendix to Statistics of Deadly Quarrels*, which appeared in print some years after his death, Richardson (1961)

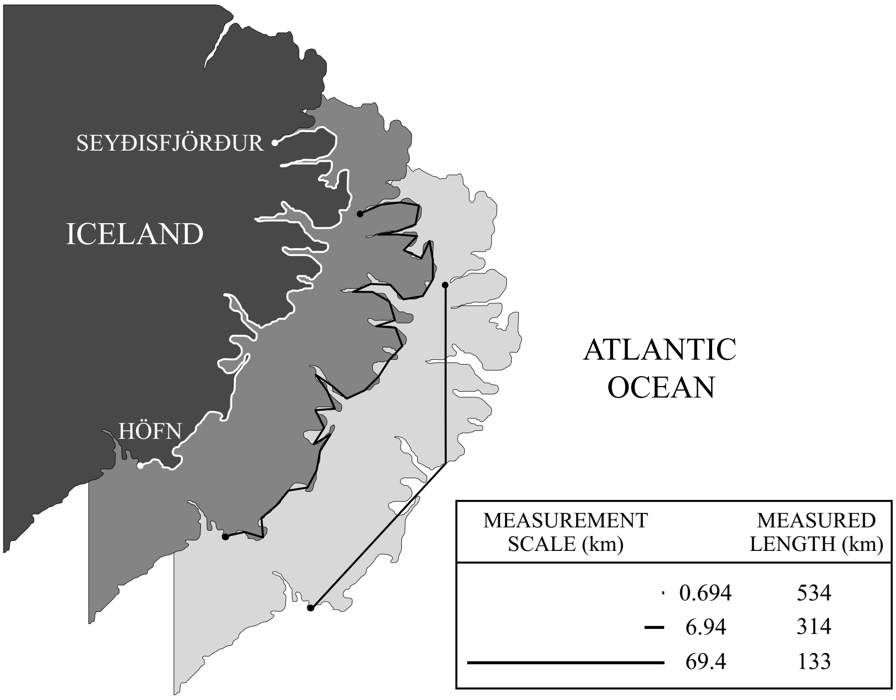


***Fig. 1.1*** The coastline of Iceland between Seyðisfjörður and Höfn, measured at three different scales. The finest-scale measurement is shown as the white curve on the dark-gray (top) map. The inset table indicates the measured coastline length for the three different scales. The finer the scale of the measurement, the greater the detail captured, and the larger the outcome for the length of the coastline.

showed that this relation takes the form of a power-law function,

$$d \propto \mathsf{s}^c, \tag{1.1}$$

where $d$ is the length of the coastline, $\mathsf{s}$ is the measurement scale, and $c$ is a (negative) power-law exponent.[1] A circle, in contrast, does not fit this form, suggesting that real coastlines do not resemble simple geometrical shapes, and that Richardson's relation is not spurious.

[1] In *Statistics of Deadly Quarrels*, Richardson (1960) had previously demonstrated that the magnitude of a war related to its frequency of occurrence by means of a power-law function. For each tenfold increase in size, he found roughly a threefold decrease in frequency. He also determined that the occurrences of wars closely follow a Poisson process (see Sec. 4.1), albeit with a quasi-periodic modulation of the rate. He further concluded that states bordering a large number of contiguous states tended to become involved in wars more often — hence, his attention to the lengths of frontiers and coastlines. A biographical sketch of Richardson is provided by Mandelbrot (1982, Chapter 40).

Not long after Richardson's work was published, Benoit Mandelbrot (1967a, 1975) revisited the issue of coastline length, and began to lay the groundwork for what would later be called fractal analysis.[2] The dependence of a measurement outcome on the scale chosen to make that measurement is the hallmark of a **fractal object**, and coastlines are indeed fractal. The power-law relationship provided in Eq. (1.1) offers a useful description of coastlines and other fractal objects, as we will see in Chapter 2.

## 1.2 POINT PROCESSES

After a long day measuring coastlines, our geographer drives home for the night. Unfortunately, many other people have chosen this same hour to drive, and our geographer encounters traffic. Since this is a recurring problem, the local government has decided to charter a study of the traffic flow patterns along the road from the coast. What is the best method for describing the traffic?

Certain details of the vehicles, such as their color or the number of occupants, are irrelevant to the traffic flow. To first order, a listing of the times at which a vehicle crosses a given point on the road provides the most salient information. Such a record, in the form of a set of marks on a line, is called a **point process**.[3] The mathematical theory of point processes forms a surprisingly rich field of study despite its seeming simplicity.[4]

Figure 1.2 illustrates the process of reducing a moving set of vehicles on a roadway to a point process. The figure comprises snapshots of the same stretch of single-lane roadway, at different times as successive vehicles pass a fixed measurement location (indicated by vertical dashed line). Vehicles yet to reach the measurement location are shown as white. As each vehicle passes this location, it turns light gray, and the point-process record at the right accrues a corresponding mark at that time, indicating the passing event. Many of the vehicles (labeled by letters) appear in several of the snapshots. Dark gray indicates a vehicle that passed the dashed line elsewhere in the figure whereas black indicates a vehicle that passed this location yet earlier.

## 1.3 FRACTAL-BASED POINT PROCESSES

This book concerns **fractal-based point processes** — processes with fractal properties comprised of discrete events, either identical or taken to be identical.

[2] A photograph of Mandelbrot stands at the beginning of Chapter 7. A recent interview by Olson (2004) offers some personal reminiscences about his life and career.
[3] A point process is sometimes called a **time series**, although this latter designation usually refers to a discrete-time process.
[4] Point processes relevant to vehicular traffic flow have been studied, for example, by Chandler, Herman & Montroll (1958); Komenani & Sasaki (1958); Bartlett (1963, 1972); Newell & Sparks (1972); Bovy (1998); and Kerner (1998, 1999).

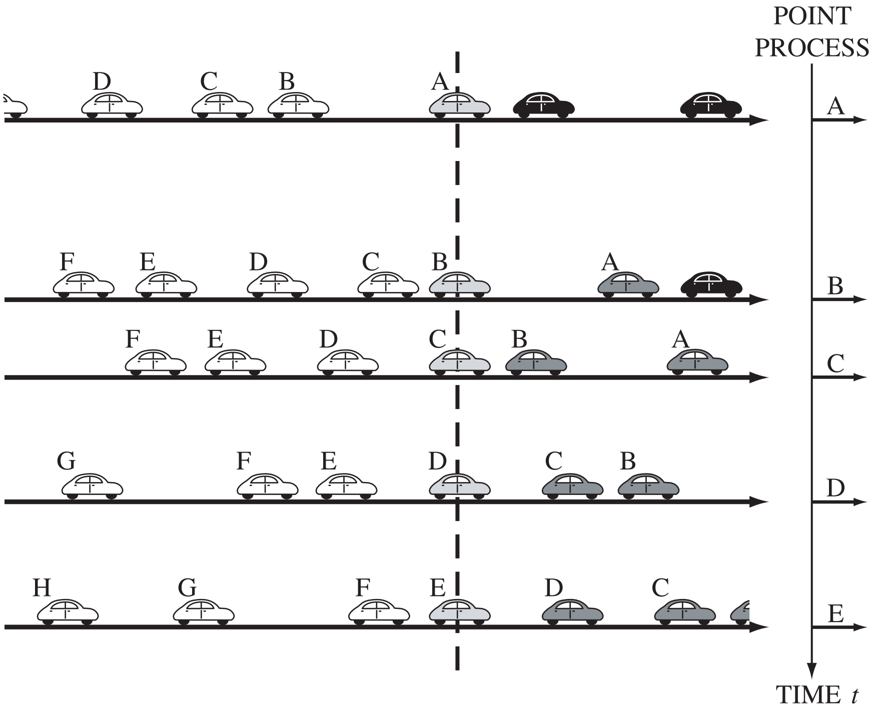


***Fig. 1.2*** Generation of a point process from vehicular traffic. Each horizontal line depicts vehicles traveling along the same stretch of single-lane road, but at different times. As each successive vehicle passes a measurement location (indicated by the vertical dashed line), it turns light gray in color and a corresponding mark appears in the final point-process representation (short horizontal arrow at far right), denoting the occurrence of an event at that time. In each depiction of the roadway, vehicles that have not yet passed the dashed line are shown as white, whereas those that already passed it elsewhere in the figure appear as dark gray. Vehicles shown in black passed the dashed line at a yet earlier time.

A more detailed introduction to fractals, and their connection to chaos, is provided in *Chapter 2*. We define point processes, and consider measures useful for characterizing them, in *Chapter 3*. *Chapter 4* sets forth a number of important examples of point processes. With an understanding of fractals and point processes in hand, we address their integration in *Chapter 5*. Mathematical formulations for several important fractal-based point-process families follow, in *Chapters 6–10*. An exposition of how various operations affect these processes appears in *Chapter 11*. *Chapter 12* considers techniques for the analysis and estimation of fractal-based point processes. Finally, *Chapter 13* is devoted to computer network traffic, an important application that serves as an illustration of the various approaches and models set forth in earlier chapters.

***Problems***

**1.1** *Length of Icelandic coastline at different measurement scales* Table 1.1 provides results for the length of a portion of the east coast of Iceland, measured between the towns of Seyðisfjörður and Höfn, at different measurement scales.

| Measurement Scale s (km) | Number of Segments | Coastline Length $d$ (km) |
|---|---|---|
| 0.694 | 769 | 534 |
| 1.39 | 306 | 425 |
| 2.78 | 140 | 389 |
| 6.94 | 45.2 | 314 |
| 13.9 | 20.3 | 282 |
| 27.8 | 6.33 | 176 |
| 50.0 | 2.84 | 142 |
| 69.4 | 1.92 | 133 |

***Table 1.1*** Length of the Icelandic coastline between the towns of Seyðisfjörður and Höfn, determined using eight different measurement scales. The finer the scale, the greater the length. These measurements were made from a map with a resolution of 0.694 km, as determined by the edge length of the minimum pixel size. The point-to-point distance between the two towns is 125 km.

**1.1.1.** Plot the coastline length $d$ vs. the measurement scale s on doubly logarithmic coordinates.

**1.1.2.** Use the plot generated in Prob. 1.1.1, together with Eq. (1.1), to determine the power-law exponent $c$ that characterizes the eastern Icelandic coastline.

**1.1.3.** Using this same form of analysis, Richardson (1961, p. 169) showed that Eq. (1.1) indeed characterized several coastlines. He reported the following results: (1) $c \approx -0.02$ for the South African coastline between Swakopmund and Cape Santa Lucia; (2) $c \approx -0.13$ for the Australian coastline; and (3) $c \approx -0.25$ for the west coast of Great Britain. Compare the scaling exponent $c$ you obtain for the east coast of Iceland with those determined by Richardson. Consult an atlas to estimate the relative roughness of the four coastlines and relate this to the power-law exponents $c$.

**1.2** *Circles and fractals* Suppose that we calculate entries for a table similar to the ones that appear in Fig. 1.1 and Table 1.1, but for a circle of unit circumference. To measure the circumference with $n$ equal line segments, inscribe a regular polygon of $n$ sides into the circle, and calculate the perimeter of the polygon. This procedure yields a perimeter that increases as the polygon side length decreases, as shown in Table 1.2.

| Polygon Side Length | Number of Sides | Polygon Perimeter |
|---|---|---|
| 0.033 | 30 | 0.998 |
| 0.098 | 10 | 0.984 |
| 0.187 | 5 | 0.935 |
| 0.276 | 3 | 0.827 |

***Table 1.2*** Polygon approximation for the perimeter of a circle.

**1.2.1.** Calculate an exact expression for the side length and total estimated perimeter as a function of the number of sides, and verify that the perimeter monotonically increases with the number of sides.

**1.2.2.** Is a circle a fractal? Why?